Statistically Secure Sigma Protocols with Abort

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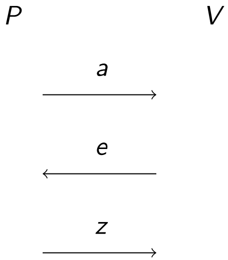
Overview

Sigma Protocols

P claims that he know some piece of information such as a secret key to a given public key.

A sigma protocol implies:

- an identification scheme.
- a signature scheme.
- a zero-knowledge protocol.
- a commitment scheme.



Security of Sigma Protocols

The security of a sigma protocol is based on the hardness of some computational problem such as:

- Prime factorization: Given $n = p \cdot q$, find the primes p and q.
- Discrete logarithm: Given $h = g^w \bmod p$, find w.

But, what about lattice problems such as the shortest vector problem (SVP)?

- Given a lattice $\hat{v}$, find the shortest vector $\vec{v}$ in $\hat{v}$.
- SVP reduces to the problem of finding *small* preimages.
- And hence, traditionally sigma protocols are insure when using lattice problems.

Setup of Protocol 4.1 (1/2)

- A polynomial time bounded prover P and verifier V.
- An additive homomorphic function $f : (\mathbb{Z}^n, +) \mapsto (G, \circ)$ such that $f(\vec{c} + \vec{d}) = f(\vec{c}) \circ f(\vec{d})$ for all $\vec{c}, \vec{d} \in \mathbb{Z}^n$.
- The interval $I = [-(S \cdot B - B); S \cdot B - B]$ for $S, B \geq 1$.
- The witness $\vec{w} \in \mathbb{Z}^n$ for the problem x in the relation R where $\|\vec{w}\|_\infty \leq B$, $x = (f, y)$ and $y = f(\vec{w})$.
- The commitment scheme commit with public key pk, which comes in two flavors:
  - Unconditional binding and computational hiding.
  - Computational binding and perfect hiding.
- The provers abort probability $\Pr[\vec{z} \notin I^n] = 1 - \left(\frac{2 \cdot (S \cdot B - B) + 1}{2 \cdot (S \cdot B) + 1}\right)^n$.

Setup of Protocol 4.1 (2/2)

- The limit $E = t \cdot (1 - \Pr[\vec{z} \notin I^n]) - t \cdot \epsilon$ where $\epsilon \in (0; 1]$.
- The linear secret sharing code $\mathrm{C} = [n + \ell, k, d]_q$ that satisfies:
  - $(d^\perp - \ell - 1)$-privacy where $d^\perp$ is the minimum distance of the dual code $\mathrm{C}^\perp$.

Massey's LSSS: To secret share $s \in \mathbb{F}_q^\ell$ we choose $c = (c_1, \ldots, c_\ell, c_{\ell+1}, \ldots, c_{\ell+n}) \in_R C$ such that $s = (c_1, \ldots, c_\ell)$ where $(c_{\ell+1}, \ldots, c_{\ell+n})$ are the shares of s and $|C| = q^k$. And hence, for Protocol 4.1 we choose:

- $\ell = 1$ for small codewords
- a large k to increase the number of codewords
- an E such that $d > 2 \cdot (t - E)$ where $t = n + \ell$

Protocol 4.1 (1/2)

Prover $P(\vec{w}, x)$ | **Verifier** $V(x)$

$\vec{r_i} \in_R \mathbb{Z}^n$ such that

$\|\vec{r_i}\|_\infty \leq S \cdot B$

$a_i = f(\vec{r_i})$

$s_i \in_R \mathbb{Z}$

$com_i = \text{commit}_{\text{pk}}(a_i, s_i)$

$$\xrightarrow{(com_1, \ldots, com_t)}$$

$$\xleftarrow{e \in_R \{0,1\}^k}$$

Protocol 4.1 (2/2)

$c = \mathrm{C}(e)$

$\vec{z_i} = \vec{r_i} + c \cdot \vec{w}$

if $\vec{z_i} \in I^n$ then

$\mathcal{Z}_i = (\vec{z_i}, a_i, s_i)$

else $\mathcal{Z}_i = \perp$

$$\xrightarrow{(\mathcal{Z}_1, \ldots, \mathcal{Z}_t)}$$

$c = \mathrm{C}(e)$

accept iff at least E :

$\mathcal{Z}_i \neq \perp$,

$com_i = \mathrm{commit}_{\mathrm{pk}}(a_i, s_i)$

and $f(\vec{z_i}) = a_i \circ y^c$

Theorem (4.2)

Let $\mathsf{commit}^{ub,ch}$ *be an unconditional binding and computational hiding commitment scheme and* $\mathsf{commit}^{cb,ph}$ *a computational binding and perfect hiding commitment scheme.*

Protocol 4.1 satisfies

	$\mathsf{commit}^{ub,ch}$	$\mathsf{commit}^{cb,ph}$
Completeness	*Statistical*	*Statistical*
Special soundness	*Perfect*	*Statistical*
sHVZK	*Computational*	*Perfect*

and hence is a statistically secure sigma protocol.

Theorem (3.1)

Let $\mathsf{commit}^{ub,ch}$ *be an unconditional binding and computational hiding commitment scheme and* $\mathsf{commit}^{cb,ph}$ *a computational binding and perfect hiding commitment scheme.*

The general framework with abort (Protocol 3.1) satisfies

	$\mathsf{commit}^{ub,ch}$	$\mathsf{commit}^{cb,ph}$
Completeness	*Aborts with prob.* $\Pr[\vec{z} \notin I^n]$	
Special soundness	*Perfect*	*Statistical*
sHVZK	*Computational*	*Perfect*

Proof of Theorem 4.2

Let (P, V) be the general framework with abort and let (P_Σ, V_Σ) be Protocol 4.1.

Statistical Completeness (1/6)

Definition

If P_Σ and V_Σ follows the protocol on input x and private input $\vec{w}$ to P_Σ where $(\vec{w}, x) \in R$, then is the probability that V_Σ outputs reject negligible in t.

Statistical Completeness (2/6)

Proof.

Assume that P_Σ know a witness $\vec{w}$ such that $(\vec{w}, x) \in R$.

We have to prove, that the following limit E implies that V_Σ only rejects P_Σ with probability negligible in t.

$$E = t \cdot (1 - \Pr[\vec{z} \notin I^n]) - t \cdot \epsilon$$

Statistical Completeness (3/6)

A conversation is on the form $(com_i, c, \mathcal{Z}_i)$ for $i = 1, \ldots, t$ where:

- $(com_1, \ldots, com_t)$ and $(\mathcal{Z}_1, \ldots, \mathcal{Z}_t)$ are fully independent because of the used randomness.
  - $com_i = \text{commit}_{\text{pk}}(a_i, s_i)$
  - $\mathcal{Z}_i = \perp$ or $\mathcal{Z}_i = (\vec{z}_i, a_i, s_i)$
- c is only $(d^\perp - 2)$-wise independent because of the linear secret sharing code C.
  - $c = \mathrm{C}(e)$

Statistical Completeness (4/6)

We can use the Chernoff-Hoeffding bound with limited independence (CHwLI).

1. Let X_i for $i = 1, \ldots, t$ denote the conversations where:
   - $X_i = 1$ if conversation i is an accepting conversation.
   - $X_i = 0$ otherwise.
2. Define $X = \sum_{i=1}^{t} X_i$ and $\mu(t) = t \cdot (1 - \Pr[\vec{z} \notin I^n])$.
3. Let $d^{\perp} = t \cdot \alpha$ for some $\alpha \in [0; 1]$.
4. Define the independence as $\ell(t) = (t \cdot \alpha) - 2$.

Statistical Completeness (5/6)

CHwLI says that

$$\Pr[|X - \mu(t)| \geq \epsilon \cdot \mu(t)]$$

is negligible in t for any $\ell(t)$ where ϵ is the same as in E.

1. Use CHwLI to argue that X lies between 1 and $\mu(t) - \epsilon \cdot \mu(t)$ with probability negligible in t.
2. Prove that $|E - \mu(t)| \geq \epsilon \cdot \mu(t)$.

Statistical Completeness (6/6)

$$
\begin{aligned}
|E - \mu(t)| &= |(t \cdot (1 - \Pr[\vec{z} \notin I^n]) - t \cdot \epsilon) - \mu(t)| \\
&= |(\mu(t) - t \cdot \epsilon) - \mu(t)| \\
&= |-t \cdot \epsilon| \\
&= t \cdot \epsilon \\
&\geq \mu(t) \cdot \epsilon
\end{aligned}
$$

□

Statistical Special Soundness (1/3)

Definition

Let $(com, c, \mathcal{Z})$ and $(com', c', \mathcal{Z}')$ be two accepting conversations for the same x where $c \neq c'$. Furthermore, let Ext be a probabilistic polynomial time knowledge extractor. The probability that Ext on input $(x, com, com', c, c', \mathcal{Z}, \mathcal{Z}')$ *can't* extract a correct witness from the prover is negligible in the length of x.

Statistical Special Soundness (2/3)

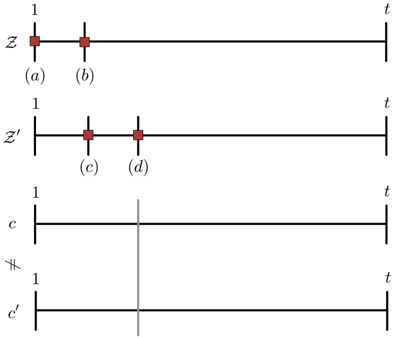
Proof.

Let $com = (com_1, \ldots, com_t)$ and $\mathcal{Z} = (\mathcal{Z}_1, \ldots, \mathcal{Z}_t)$.

1. Assume that P_Σ can produce two accepting conversations $(com, c, \mathcal{Z})$ and $(com', c', \mathcal{Z}')$ with different challenges $c \neq c'$ for $(\mathsf{P}_\Sigma, \mathsf{V}_\Sigma)$.
2. Prove that there exists an index j such that $(com_j, c_j, \mathcal{Z}_j)$ and $(com'_j, c'_j, \mathcal{Z}'_j)$ are two accepting conversations with different challenges $c_j \neq c'_j$ for (P, V).
3. Since (P, V) satisfies statistical special soundness, we have that $(\mathsf{P}_\Sigma, \mathsf{V}_\Sigma)$ also satisfies this property.

Statistical Special Soundness (3/3)

- At most $t - E$ aborting conversations.
- $\mathcal{Z}_i = \bot$ for all i between point (a) and (b).
- $\mathcal{Z}'_i = \bot$ for all i between point (c) and (d).
- Make sure that $\Delta(c, c') > 2 \cdot (t - E)$ for all $c, c' \in \mathrm{C}$ by choosing $d > 2 \cdot (t - E)$.



□

Computational sHVZK

Definition

There exists a probabilistic polynomial time simulator Sim, which on input x and a random challenge c, outputs an accepting conversation $(com, c, \mathcal{Z})$ such that $\mathrm{Sim}(x, c) \sim^c (\mathrm{P}_\Sigma(\vec{w}), \mathrm{V}_\Sigma)(x)$.

Proof.

Since (P, V) satisfies computational sHVZK, we have that $(\mathrm{P}_\Sigma, \mathrm{V}_\Sigma)$ also satisfies this property because sHVZK is invariant under parallel composition.

$\square$

Conclusion

We have constructed a *statistically secure sigma protocol* that satisfies:

	$\text{commit}^{ub,ch}$	$\text{commit}^{cb,ph}$
Completeness	Statistical	Statistical
Special soundness	Perfect	Statistical
sHVZK	Computational	Perfect

and where we can base the security on:

- The prime factorization problem.
- The discrete logarithm problem.
- Lattice problems such as the shortest vector problem.